ON EXISTENCE AND STABILITY OF SOLUTIONS OF STEADY-STATE THEORY OF A THERMAL EXPLOSION

(O SUSHCHESTVOVANII I USTOICHIVOSTI RESHENII V STATSIONARNOI TEORII TEPLOVOGO VZRYVA)

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Theory of thermal explosion in a vessel with exothermic reactions was developed in papers [1 to 3]. Cases of plane, cylindrical and spherical symmetry were examined. Investigation of stability of equations of steady-state theory of thermal explosion [4] showed that only one regime will be stable which corresponds to the lower temperature which can establish itself in the vessel.

A qualitative investigation of stability of solutions of steady-state theory of thermal explosions is carried out below for bounded vessels.

In the review article of [5] the problem was posed to prove that from the existence of a steady-state solution for a certain vessel, the existence of a solution follows for a vessel inserted in it. In this paper a proof is given for this statement, which is different from the one presented in [6]. It is also proved in this paper that from the existence of a stable solution for a vessel there follows the existence of a stable solution for an inserted vessel.

1. 1°. Stationary theory of thermal explosion leads to the problem of Dirichlet:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\varphi(u)=0, \qquad u|_\Gamma=0 \tag{1.1}$$

Here u is dimensionless temperature, x, y, z are dimensionless coordinates, Γ is the boundary of the region, $\varphi(u)$ is the function of heat generation. It is assumed that

$$\varphi \geqslant 0, \quad d\varphi / du \geqslant 0, \quad d^2\varphi / du^2 \geqslant 0 \quad (u \geqslant 0) \tag{1.2}$$

Usually the expression $\varphi = e^u$ is taken for the function φ. This expression is obtained through some simplification of relationships of chemical kinetics. Solution of problem (1.1) is equivalent to the solution of nonlinear integral equation

$$u(\mathrm{x}) = \frac{1}{\pi}\int_\sigma G(\mathrm{x}, \mathrm{y})\,\varphi(u(\mathrm{y}))\,d\mathrm{y} \tag{1.3}$$

Here x, y are points of domain σ, $G(\mathrm{x}, \mathrm{y})$ is Green's functions of the Laplace operator for the Dirichlet problem. (The absence of concentrated heat sources in the vessel is assumed).

2°. Let us examine the problem of existence of solution of Equation (1.3) for the domain $\sigma' \subseteq \sigma$ under the condition of existence of solution of Equation (1.3) for the domain σ.

We introduce the following notation for the operators:

$$Au = \frac{1}{\pi}\int_{\sigma} G(\mathbf{x}, \mathbf{y})\,\varphi(u)\,d\mathbf{y}, \qquad A'u = \frac{1}{\pi}\int_{\sigma'} G'(\mathbf{x}, \mathbf{y})\,\varphi(u)\,d\mathbf{y} \tag{1.4}$$

Let u_0 be a solution of Equation

$$Au = u \tag{1.5}$$

We shall show that in such a case a convex closed set $0 \leq u \leq u_0$ turns into itself under transformation A. In fact $0 \leq Au < Au_0 = u_0$, because $d\varphi/du \geq 0$. The set of functions $0 \leq u \leq u_0$ is determined on σ.

Let us examine this set in the domain $\sigma' \subseteq \sigma$. We shall show that the operator $A'u$ transforms the set $0 \leqslant u \leqslant u_0\,(\mathbf{x} \in \sigma')$ into itself. For this it is sufficient to show that

$$\int_{\sigma' \subseteq \sigma} G'\varphi\,d\sigma \leqslant \int_{\sigma' \subseteq \sigma} G\varphi\,d\sigma \quad \text{or} \quad G' \leqslant G \tag{1.6}$$

Here

$$G(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}, \mathbf{y}) + g(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \sigma$$

Here $P(\mathbf{x}, \mathbf{y})$ is the potential of a point source in the free space, while $g(\mathbf{x}, \mathbf{y})$ is a harmonic function in σ such that $G|_{\Gamma} = 0$. From fundamental properties of Green's function it follows that $G' \leq G$ or $g' \leq g$ on the boundary Γ' of domain $\sigma' \subseteq \sigma$.

Since g and g' are harmonic functions, then $g' \leq g$ for all $\mathbf{x}, \mathbf{y} \in \sigma'$. From this $G' \leq G$ on σ.

Thus, the completely continuous operator $A'u$ transforms the set $0 \leq u \leq u_0$ into itself. From this it follows according to the theorem of Leray and Schauder [7] that representation of A' has a stationary point. In the domain σ'

$$u_0' \leqslant u_0 \tag{1.7}$$

2. 1°. The investigation of stability of solutions in the steady-state theory of thermal explosion will be carried out by the method of small perturbations.

Let us examine the nonsteady-state equation of heat transfer

$$\frac{\partial u}{\partial t} = \Delta u + \varphi(u), \qquad u|_{\Gamma} = 0 \tag{2.1}$$

The solution of this equation is presented in the form

$$u(\mathbf{x}, \mathbf{y}) = u_0(\mathbf{x}) + \omega(\mathbf{x}, t) \tag{2.2}$$

where u_0 is the solution of he problem (1.1) the stability of which is being examined, and $\omega(\mathbf{x}, t)$ is a small perturbation. The initial and boundary conditions for $\omega(\mathbf{x}, t)$ have the form

$$\omega|_{t=0} = \varepsilon(\mathbf{x}), \qquad \omega|_{\Gamma} = 0, \qquad (\varepsilon|_{\Gamma} = 0) \tag{2.3}$$

Taking into consideration the smallness of $\omega(\mathbf{x}, t)$ we obtain from (2.1) and (2.2) the following equation:

$$\frac{\partial \omega}{\partial t} = \Delta\omega + \frac{d\varphi(u_0)}{du}\,\omega \tag{2.4}$$

We shall seek the solution of this equation in the form of a series

$$\omega = \sum_{i=0}^{\infty} a_i f_i(\mathbf{x})\,e^{-\lambda_i t}, \qquad f_i|_{\Gamma} = 0 \tag{2.5}$$

It will be required that the function $f_i(x)\,e^{-\lambda_i t}$ satisfy (2.4), then we will obtain

$$\Delta f_i + \left(\frac{d\varphi(u_0)}{du} + \lambda_i\right) f_i = 0, \qquad f_i|_{\Gamma} = 0 \tag{2.6}$$

In this manner the problem has been reduced to a problem of the Sturm-Liouvilie type (multidimensional analog). The initial condition can always be satisfied because it is known that the system of eigenfunctions of this problem is complete.

If in the spectrum of eigenvalues all $\lambda_i > 0$, then the solution u_0 under examination is stable; if however even one $\lambda_i < 0$ is found, the solution is unstable. From the theory of boundary value problems of the examined type, it is known that $\lambda_0 \leq \lambda_i$. Therefore it is sufficient to know only the sign of λ_0 to answer the question of stability.

2°. In paper [9] it was proved that for the existence of solution of problem (1.1), which satisfies the condition

$$0 \leqslant u \leqslant R \qquad (R = \text{const})$$

it is sufficient to satisfy the condition

$$D \leqslant \varkappa^2 \frac{R}{\varphi(R)} \tag{2.7}$$

Here D is the diameter of the domain σ, and $\varkappa$ is a constant depending on the dimensionality of σ.

It follows from (2.7) that for sufficiently small D the problem (1.1) has solutions which are arbitrarily small. We shall demonstrate the stability of these solutions in the framework of the theory of small perturbations. As a preliminary step let us formulate two theorems the proof of which can be found in [8].

Equation $\Delta f(\mathbf{x}) + (\lambda + q(\mathbf{x}))f(\mathbf{x}),\ \mathbf{x} \in E$ is examined. The boundary condition is that the function $f(\mathbf{x})$ becomes zero on the boundary E.

Theorem 1. If eigenvalues of the problem with a function $q(\mathbf{x})$, a domain E and a boundary condition $f = 0$ are compared to eigenvalues of a problem in which the domain and the boundary conditions are preserved, while $q(\mathbf{x})$ is replaced by another function, then every eigenvalue does not increase with increasing $q(\mathbf{x})$.

Theorem 2. If eigenvalues of this problem are compared with eigenvalues of a problem in which the domain E is replaced by domain $E' \subseteq E$, the condition $f = 0$ is established on the boundary of domain E' and functions $q(\mathbf{x})$ is retained the same as before, then the eigenvalues do not increase for an increase in the domain.

We shall examine the problem

$$\Delta f_i + \left(\frac{d\varphi(u)}{du} + \lambda_i\right) f_i = 0, \qquad f_i|_\Gamma = 0 \tag{2.8}$$

Let $0 \leq u \leq R$. Since $d^2\varphi/du^2 \geq 0$, then

$$\frac{d\varphi}{du} \leqslant \frac{d\varphi(R)}{du} = \theta$$

We substitute θ for $d\varphi/du$ in Equation (2.8) and arrive at the following problem:

$$\Delta f_i + \delta_i f_i = 0, \qquad f_i|_\Gamma = 0 \qquad (\delta_i = \lambda_i + \theta) \tag{2.9}$$

It is known that for this problem $\delta_i > 0$. It will be shown that for continuous decrease of diameter of σ, δ_i tends to infinity.

For this purpose we formulate problem (2.9) for a square with side a, which contains σ. For this problem $\delta_0 = 2\pi^2/a^2$ which tends to infinity with decreasing a.

It follows from Theorem 2 that this remains valid also for domain σ, when its diameter tends to zero. For sufficiently small domain σ, δ_0 becomes greater than θ and therefore λ_0 of problem (2.9) becomes greater than zero.

Let the problem (2.8) be formulated for these small σ. Since $d\varphi/du \leq \theta$, then according to Theorem 1 the first eigenvalue of problem (2.8) is not less than the first eigenvalue of problem (2.9) and therefore it is positive.

Since for sufficiently small σ there are sufficiently small u_0, we can always assume that $0 \leq u_0 \leq R$. From this it follows that problem (2.6) has a positive spectrum for sufficiently small σ.

3°. We shall prove the existence of a stable solution of Equation (1.2) for the domain $\sigma' \subseteq \sigma$ under the condition of existence of a stable solution of Equation (1.2) in the region σ.

In Section 1 it was proved that solution u_0' exists for the region σ', and $u_0' \leq u_0$. It will be shown that u_0' is stable if u_0 is stable. Problem (2.6) will be formulated for region σ. Since u_0 is stable, $\lambda_0 > 0$. Now we shall formulate the same problem for u_0' and domain σ'. Since $\sigma' \subseteq \sigma$ and

$$\frac{d\varphi(u_0')}{du} \leqslant \frac{d\varphi(u_0)}{du}$$

(by virtue of (1.7)), we immediately have as a direct consequence of Theorem 1 and 2 that the zeroth eigenvalue corresponding to solution u_0' is positive.

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